Polynomials Problems

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1. Find all polynomial $P$ satisfying: $P(x^2 + 1) = P(x)^2 + 1$.

2. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
$$f(x^n + 2f(y)) = (f(x))^n + y + f(y) \quad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{Z}_{\geq 2}.$$

3. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that
$$x^2y^2(f(x+y) - f(x) - f(y)) = 3(x+y)f(x)f(y)$$

4. Find all polynomials $P(x)$ with real coefficients such that
$$P(x)P(x+1) = P(x^2) \quad \forall x \in \mathbb{R}.$$

5. Find all polynomials $P(x)$ with real coefficient such that
$$P(x)Q(x) = P(Q(x)) \quad \forall x \in \mathbb{R}.$$

6. Find all polynomials $P(x)$ with real coefficients such that if $P(a)$ is an integer, then so is $a$, where $a$ is any real number.

7. Find all the polynomials $f \in \mathbb{R}[X]$ such that
$$\sin f(x) = f(\sin x), \quad (\forall) x \in \mathbb{R}.$$

8. Find all polynomial $f(x) \in \mathbb{R}[x]$ such that
$$f(x)f(2x^2) = f(2x^3 + x^2) \quad \forall x \in \mathbb{R}.$$

9. Find all real polynomials $f$ and $g$, such that:
$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$
for all $x \in \mathbb{R}$.

10. Find all polynomials $P(x)$ with integral coefficients such that $P(P'(x)) = P'(P(x))$ for all real numbers $x$.

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11. Find all polynomials with integer coefficients $f$ such that for all $n > 2005$ the number $f(n)$ is a divisor of $n^{a-1} - 1$.

12. Find all polynomials with complex coefficients $f$ such that we have the equivalence: for all complex numbers $z, z \in [-1, 1]$ if and only if $f(z) \in [-1, 1]$.

13. Suppose $f$ is a polynomial in $\mathbb{Z}[X]$ and $m$ is integer. Consider the sequence $a_i$ like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials $f$ and all integers $m$ that for each $i$:

$$a_i | a_{i+1}$$

14. $P(x), Q(x) \in \mathbb{R}[x]$ and we know that for real $r$ we have $p(r) \in \mathbb{Q}$ if and only if $Q(r) \in \mathbb{Q}$ I want some conditions between $P$ and $Q$. My conjecture is that there exist rational $a, b, c$ that $aP(x) + bQ(x) + c = 0$

15. Find all polynomials $f$ with real coefficients such that for all reals $a, b, c$ such that $ab + bc + ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

16. Find all polynomials $p$ with real coefficients that if for a real $a,p(a)$ is integer then $a$ is integer.

17. $\Psi$ is a real polynomial such that if $\alpha$ is irrational then $\Psi(\alpha)$ is irrational. Prove that $\deg(\Psi) \leq 1$

18. Show that the odd number $n$ is a prime number if and only if the polynomial $T_n(x)/x$ is irreducible over the integers.

19. $P, Q, R$ are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$.

a) If $P, Q, R \in \mathbb{R}[x]$, prove that $Q$ is constant polynomial.

b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

20. Let $P$ be a polynomial such that $P(x)$ is rational if and only if $x$ is rational. Prove that $P(x) = ax + b$ for some rational $a$ and $b$.

21. Prove that any polynomial $\in \mathbb{R}[X]$ can be written as a difference of two strictly increasing polynomials.

22. Consider the polynomial $W(x) = (x - a)^k Q(x)$, where $a \neq 0$, $Q$ is a nonzero polynomial, and $k$ a natural number. Prove that $W$ has at least $k + 1$ nonzero coefficients.

23. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that the equation

$$f(x) = n$$

has at least one rational solution, for each positive integer $n$.

24. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of $f$ is 1 (the coefficient of the term having the highest degree in $f$) then $f(X^2)$ is also irreducible in the ring of integer polynomials.
25. Let \( p \) be a prime number and \( f \) an integer polynomial of degree \( d \) such that \( f(0) = 0, f(1) = 1 \) and \( f(n) \) is congruent to 0 or 1 modulo \( p \) for every integer \( n \). Prove that \( d \geq p - 1 \).

26. Let \( P(x) := x^n + \sum_{k=1}^{n} a_k x^{n-k} \) with \( 0 \leq a_n \leq a_{n-1} \leq \ldots a_2 \leq a_1 \leq 1 \). Suppose that there exists \( r \geq 1, \varphi \in \mathbb{R} \) such that \( P(re^{ix}) = 0 \). Find \( r \).

27. Let \( P \) be a polynomial with rational coefficients such that \( P^{-1}(\mathbb{Q}) \subseteq \mathbb{Q} \).

Prove that \( \deg P \leq 1 \).

28. Let \( f \) be a polynomial with integer coefficients such that \( |f(x)| < 1 \) on an interval of length at least 4. Prove that \( f = 0 \).

29. prove that \( x^n - x - 1 \) is irreducible over \( \mathbb{Q} \) for all \( n \geq 2 \).

30. Find all real polynomials \( p(x) \) such that
\[
p^2(x) + 2p(x) p\left(\frac{1}{x}\right) + p^2\left(\frac{1}{x}\right) = p(x^2) p\left(\frac{1}{x^2}\right)
\]
For all non-zero real \( x \).

31. Find all polynomials \( P(x) \) with odd degree such that
\[
P(x^2 - 2) = P^2(x) - 2.
\]

32. Find all real polynomials that
\[
p(x + p(x)) = p(x) + p(p(x))
\]

33. Find all polynomials \( P \in \mathbb{C}[X] \) such that
\[
P(X^2) = P(X)^2 + 2P(X).
\]

34. Find all polynomials of two variables \( P(x, y) \) which satisfy
\[
P(a,b)P(c,d) = P(ac + bd, ad + bc), \forall a, b, c, d \in \mathbb{R}.
\]

35. Find all real polynomials \( f(x) \) satisfying
\[
f(x^2) = f(x)f(x - 1) \forall x \in \mathbb{R}.
\]

36. Find all polynomials of degree 3, such that for each \( x, y \geq 0 \):
\[
p(x + y) \geq p(x) + p(y).
\]

37. Find all polynomials \( P(x) \in \mathbb{Z}[x] \) such that for any \( n \in \mathbb{N} \), the equation \( P(x) = 2^n \) has an integer root.
38. Let $f$ and $g$ be polynomials such that $f(Q) = g(Q)$ for all rationals $Q$. Prove that there exist reals $a$ and $b$ such that $f(X) = g(aX + b)$, for all real numbers $X$.

39. Find all positive integers $n \geq 3$ such that there exists an arithmetic progression $a_0, a_1, \ldots, a_n$ such that the equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ has $n$ roots setting an arithmetic progression.

40. Given non-constant linear functions $p_1(x), p_2(x), \ldots, p_n(x)$. Prove that at least $n - 2$ of polynomials $p_1 p_2 \cdots p_{n-1} + p_n, p_1 p_2 \cdots p_{n-2} p_n + p_{n-1}, \ldots, p_2 p_3 \cdots p_n + p_1$ have a real root.

41. Find all positive real numbers $a_1, a_2, \ldots, a_k$ such that the number $a_1^{1/k} + \cdots + a_k^{1/k}$ is rational for all positive integers $n$, where $k$ is a fixed positive integer.

42. Let $f, g$ be real non-constant polynomials such that $f(Z) = g(Z)$. Show that there exists an integer $A$ such that $f(X) = g(A + x)$ or $f(x) = g(A - x)$.

43. Does there exist a polynomial $f \in \mathbb{Q}[x]$ with rational coefficients such that $f(1) \neq -1$, and $x^n f(x) + 1$ is a reducible polynomial for every $n \in \mathbb{N}$?

44. Suppose that $f$ is a polynomial of exact degree $p$. Find a rigorous proof that $S(n)$, where $S(n) = \sum_{k=0}^{n} f(k)$, is a polynomial function of (exact) degree $p + 1$ in variable $n$.

45. The polynomials $P, Q$ are such that deg $P = n$, deg $Q = m$, have the same leading coefficient, and $P^2(x) = (x^2 - 1)Q^2(x) + 1$. Prove that $P(n) = nQ(x)$.

46. Given distinct prime numbers $p$ and $q$ and a natural number $n \geq 3$, find all $a \in \mathbb{Z}$ such that the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into 2 integral polynomials of degree at least 1.

47. Let $F$ be the set of all polynomials $\Gamma$ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive integer $k$, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly $k$ distinct integer roots.

48. Find all polynomials $P(x)$ with integer coefficients such that the polynomial $Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1$ is the square of a polynomial with integer coefficients.

49. Find all polynomials $p$ with real coefficients such that for all reals $a, b, c$ such that $ab + bc + ca = 1$ we have the relation $p(a)^2 + p(b)^2 + p(c)^2 = p(a + b + c)^2$.

50. Find all real polynomials $f$ with $x, y \in \mathbb{R}$ such that $2g f(x + y) + (x - y)(f(x) + f(y)) \geq 0$. 


51. Find all polynomials such that \( P(x^3 + 1) = P((x + 1)^3) \).

52. Find all polynomials \( P(x) \in \mathbb{R}[x] \) such that \( P(x^2 + 1) = P(x)^2 + 1 \) holds for all \( x \in \mathbb{R} \).

53. Problem: Find all polynomials \( p(x) \) with real coefficients such that
\[
(x + 1)p(x - 1) + (x - 1)p(x + 1) = 2xp(x)
\]
for all real \( x \).

54. Find all polynomials \( P(x) \) that have only real roots, such that
\[
P(x^2 - 1) = P(x)P(-x).
\]

55. Find all polynomials \( P(x) \in \mathbb{R}[x] \) such that:
\[
P(x^2) + x \cdot (3P(x) + P(-x)) = (P(x))^2 + 2x^2 \quad \forall x \in \mathbb{R}
\]

56. Find all polynomials \( f, g \) which are both monic and have the same degree and
\[
f(x)^2 - f(x^2) = g(x).
\]

57. Find all polynomials \( P(x) \) with real coefficients such that there exists a polynomial \( Q(x) \) with real coefficients that satisfy
\[
P(x^2) = Q(P(x)).
\]

58. Find all polynomials \( p(x, y) \in \mathbb{R}[x, y] \) such that for each \( x, y \in \mathbb{R} \) we have
\[
p(x + y, x - y) = 2p(x, y).
\]

59. Find all couples of polynomials \( (P, Q) \) with real coefficients, such that for infinitely many \( x \in \mathbb{R} \) the condition
\[
\frac{P(x)}{Q(x)} - \frac{P(x + 1)}{Q(x + 1)} = \frac{1}{x(x + 2)}
\]
holds.

60. Find all polynomials \( P(x) \) with real coefficients, such that \( P(P(x)) = P(x)^k \) (\( k \) is a given positive integer)

61. Find all polynomials
\[
P_n(x) = n!x^n + a_{n-1}x^{n-1} + ... + a_1x + (-1)^n(n + 1)n
\]
with integers coefficients and with \( n \) real roots \( x_1, x_2, ..., x_n \), such that \( k \leq x_k \leq k + 1 \), for \( k = 1, 2, ..., n \).
62. The function $f(n)$ satisfies $f(0) = 0$ and $f(n) = n - f(f(n-1))$, $n = 1, 2, 3 \cdots$. Find all polynomials $g(x)$ with real coefficient such that

$$f(n) = [g(n)], \quad n = 0, 1, 2 \cdots$$

Where $[g(n)]$ denote the greatest integer that does not exceed $g(n)$.

63. Find all pairs of integers $a, b$ for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form $x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$

where each of $c_0, c_1, \ldots, c_{n-1}$ is equal to 1 or $-1$.

64. There exists a polynomial $P$ of degree 5 with the following property: if $z$ is a complex number such that $z^5 + 2004z = 1$, then $P(z^5) = 0$. Find all such polynomials $P$.

65. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3P(x-1) - (x-1)^3P(x+1) = 4(x^2 - 1)P(x)$$

for all real numbers $x$.

66. Find all polynomials $P(x, y)$ with real coefficients such that:

$$P(x, y) = P(x+1, y) = P(x, y+1) = P(x+1, y+1)$$

67. Find all polynomials $P(x)$ with real coefficients such that

$$(x-8)P(2x) = 8(x-1)P(x).$$

68. Find all reals $\alpha$ for which there is a nonzero polynomial $P$ with real coefficients such that

$$\frac{P(1) + P(3) + P(5) + \cdots + P(2n-1)}{n} = \alpha P(n) \quad \forall n \in \mathbb{N},$$

and find all such polynomials for $\alpha = 2$.

69. Find all polynomials $P(x) \in \mathbb{R}[X]$ satisfying

$$(P(x))^2 - (P(y))^2 = P(x+y) \cdot P(x-y), \quad \forall x, y \in \mathbb{R}.$$
72. Find all non-constant real polynomials \( f(x) \) such that for any real \( x \) the following equality holds
\[
f(\sin x + \cos x) = f(\sin x) + f(\cos x).
\]

73. Find all polynomials \( W(x) \in \mathbb{R}[x] \) such that
\[
W(x^2)W(x^3) = W(x)^5 \quad \forall x \in \mathbb{R}.
\]

74. Find all the polynomials \( f(x) \) with integer coefficients such that \( f(p) \) is prime for every prime \( p \).

75. Let \( n \geq 2 \) be a positive integer. Find all polynomials \( P(x) = a_0 + a_1 x + \cdots + a_n x^n \) having exactly \( n \) roots not greater than \( -1 \) and satisfying
\[
a_0^2 + a_1 a_n = a_n^2 + a_0 a_{n-1}.
\]

76. Find all polynomials \( P(x), Q(x) \) such that
\[
P(Q(x)) = Q(P(x)) \forall x \in \mathbb{R}.
\]

77. Find all integers \( k \) such that for infinitely many integers \( n \geq 3 \) the polynomial
\[
P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995
\]
can be reduced into two polynomials with integer coefficients.

78. Find all polynomials \( P(x), Q(x), R(x) \) with real coefficients such that
\[
\sqrt{P(x)} - \sqrt{Q(x)} = R(x) \quad \forall x \in \mathbb{R}.
\]

79. Let \( k = \sqrt{3} \). Find a polynomial \( p(x) \) with rational coefficients and degree as small as possible such that \( p(k + k^2) = 3 + k \). Does there exist a polynomial \( q(x) \) with integer coefficients such that \( q(k + k^2) = 3 + k \)?

80. Find all values of the positive integer \( m \) such that there exists polynomials \( P(x), Q(x), R(x, y) \) with real coefficient satisfying the condition: For every real numbers \( a, b \) which satisfying \( a^n - b^2 = 0 \), we always have that \( P(R(a, b)) = a \) and \( Q(R(a, b)) = b \).

81. Find all polynomials \( p(x) \in \mathbb{R}[x] \) such that \( p(x^{2008} + y^{2008}) = (p(x))^{2008} + (p(y))^{2008} \), for all real numbers \( x, y \).

82. Find all Polynomials \( P(x) \) satisfying \( P(x)^2 - P(x^2) = 2x^4 \).

83. Find all polynomials \( p \) of one variable with integer coefficients such that if \( a \) and \( b \) are natural numbers such that \( a + b \) is a perfect square, then \( p(a) + p(b) \) is also a perfect square.

84. Find all polynomials \( P(x) \in \mathbb{Q}[x] \) such that
\[
P(x) = P \left( \frac{-x + \sqrt{3 - 3x^2}}{2} \right) \quad \text{for all} \quad |x| \leq 1.
\]
85. Find all polynomials \( f \) with real coefficients such that for all reals \( a, b, c \) such that \( ab + bc + ca = 0 \) we have the following relations

\[
f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).
\]

86. Find all polynomials \( P(x, y) \) such that for all reals \( x, y \) we have

\[
P(x^2, y^2) = \frac{(x + y)^2}{2}, \frac{(x - y)^2}{2}.
\]

87. Let \( n \) and \( k \) be two positive integers. Determine all monic polynomials \( f \in \mathbb{Z}[X] \), of degree \( n \), having the property that \( f(n) \) divides \( f(2^k \cdot a) \), for all \( a \in \mathbb{Z} \), with \( f(a) \neq 0 \).

88. Find all polynomials \( P(x) \) such that

\[
P(x^2 - y^2) = P(x + y)P(x - y).
\]

89. Let \( f(x) = x^4 - x^3 + 8ax^2 - ax + a^2 \). Find all real number \( a \) such that \( f(x) = 0 \) has four different positive solutions.

90. Find all polynomial \( P \in \mathbb{R}[x] \) such that: \( P(x^2 + 2x + 1) = (P(x))^2 + 1 \).

91. Let \( n \geq 3 \) be a natural number. Find all nonconstant polynomials with real coefficients \( f_1(x), f_2(x), \ldots, f_n(x) \), for which

\[
f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), \quad 1 \leq k \leq n,
\]

for every real \( x \) (with \( f_{n+1}(x) \equiv f_1(x) \) and \( f_{n+2}(x) \equiv f_2(x) \)).

92. Find all integers \( n \) such that the polynomial \( p(x) = x^n - nx - n - 2 \) can be written as product of two non-constant polynomials with integral coefficients.

93. Find all polynomials \( p(x) \) that satisfy

\[
(p(x))^2 - 2 = 2p(2x^2 - 1) \quad \forall x \in \mathbb{R}.
\]

94. Find all polynomials \( p(x) \) that satisfy

\[
(p(x))^2 - 1 = 4p(x^2 - 4x + 1) \quad \forall x \in \mathbb{R}.
\]

95. Determine the polynomials \( P \) of two variables so that:

a.) for any real numbers \( t, x, y \) we have \( P(tx, ty) = t^n P(x, y) \) where \( n \) is a positive integer, the same for all \( t, x, y \);

b.) for any real numbers \( a, b, c \) we have \( P(a + b, c) + P(b + c, a) + P(c + a, b) = 0 \);

c.) \( P(1, 0) = 1 \).

96. Find all polynomials \( P(x) \) satisfying the equation

\[
(x + 1)P(x) = (x - 2010)P(x + 1).
\]
97. Find all polynomials of degree 3 such that for all non-negative reals $x$ and $y$ we have

$$p(x + y) \leq p(x) + p(y).$$

98. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a)$$

for all $a, b, c \in \mathbb{R}$.

99. Find all polynomials $P(x)$ with real coefficients such that

$$P(x^2 - 2x) = (P(x - 2))^2$$

100. Find all two-variable polynomials $p(x, y)$ such that for each $a, b, c \in \mathbb{R}$:

$$p(ab, c^2 + 1) + p(bc, a^2 + 1) + p(ca, b^2 + 1) = 0.$$
Solutions