Geometry Marathon

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1. Inradius of a triangle, with integer sides, is equal to 1. Find the sides of the triangle and prove that one of its angle is $90^\circ$.

2. Let $O$ be the circumcenter of an acute triangle $ABC$ and let $k$ be the circle with center $S$ that is tangent to $O$ at $A$ and tangent to side $BC$ at $D$. Circle $k$ meets $AB$ and $AC$ again at $E$ and $F$ respectively. The lines $OS$ and $ES$ meet $k$ again at $I$ and $G$. Lines $BO$ and $IG$ intersect at $H$. Prove that $GH = \frac{DF^2}{AF}$.

3. $ABCD$ is parallelogram and a straight line cuts $AB$ at $\frac{AB}{x}$ and $AD$ at $\frac{AD}{x}$ and $AC$ at $x \cdot AC$. Find $x$.

4. In $\triangle ABC$, $\angle BAC = 120^\circ$. Let $AD$ be the angle bisector of $\angle BAC$. Express $AD$ in terms of $AB$ and $BC$.

5. In a triangle $ABC$, $AD$ is the feet of perpendicular to $BC$. The inradii of $ADC$, $ADB$ and $ABC$ are $x$, $y$, $z$. Find the relation between $x$, $y$, $z$.

6. Prove that the third pedal triangle is similar to the original triangle.

7. $ABCDE$ is a regular pentagon and $P$ is a point on the minor arc $AB$. Prove that $PA + PB + PD = PC + PE$.

8. Two congruent equilateral triangles, one with red sides and one with blue sides overlap so that their sides intersect at six points, forming a hexagon. If $r_1$, $r_2$, $r_3$, $b_1$, $b_2$, $b_3$ are the red and blue sides of the hexagon respectively, prove that
   (a) $r_1^2 + r_2^2 + r_3^2 = b_1^2 + b_2^2 + b_3^2$
   (b) $r_1 + r_2 + r_3 = b_1 + b_2 + b_3$

9. if in a quadrilateral $ABCD$, $AB + CD = BC + AD$. Prove that the angle bisectors are concurrent at a point which is equidistant from the sides of the sides of the quadrilateral.

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10. In a triangle with sides \(a, b, c\), let \(r\) and \(R\) be the inradius and circumradius respectively. Prove that for all such non-degenerate triangles,

\[
2rR = \frac{abc}{a + b + c}
\]

11. Prove that the area of any non degenerate convex quadrilateral in the cartesian plane which has an incircle is given by \(\Delta = rs\) where \(r\) is the inradius and \(s\) is the semiperimeter of the polygon.

12. Let \(ABC\) be a equilateral triangle with side \(a\). \(M\) is a point such that \(MS = d\), where \(S\) is the circumcenter of \(ABC\). Prove that the area of the triangle whose sides are \(MA, MB, MC\) is

\[
\frac{\sqrt{3}|a^2 - 3d^2|}{12}
\]

13. Prove that in a triangle,

\[
SI_i^2 = R^2 + 2Rr_a
\]

14. Find the locus of \(P\) in a triangle if \(PA^2 = PB^2 + PC^2\).

15. In an acute triangle \(ABC\), let the orthocenter be \(H\) and let its projection on the median from \(A\) be \(X\). Prove that \(BHXC\) is cyclic.

17. If \(ABC\) is a right triangle with \(A = 90^\circ\), if the incircle meets \(BC\) at \(X\), prove that \([ABC] = BX \cdot XC\).

18. \(n\) regular polygons in a plane are such that they have a common vertex \(O\) and they fill the space around \(O\) completely. The \(n\) regular polygons have \(a_1, a_2, \cdots, a_n\) sides not necessarily in that order. Prove that

\[
\sum_{i=1}^{n} \frac{1}{a_i} = \frac{n - 2}{2}
\]

19. Let the equation of a circle be \(x^2 + y^2 = 100\). Find the number of points \((a, b)\) that lie on the circle such that \(a\) and \(b\) are both integers.

20. \(S\) is the circumcentre of the \(\triangle ABC\). \(\triangle DEF\) is the orthic triangle of \(\triangle ABC\). Prove that \(SA\) is perpendicular to \(EF\), \(SB\) is the perpendicular to \(DF\) and \(SC\) is the perpendicular to \(DE\).

21. \(ABCD\) is a parallelogram and \(P\) is a point inside it such that \(\angle APB + \angle CPD = 180^\circ\). Prove that

\[
AP \cdot CP + BP \cdot DP = AB \cdot BC
\]
22. \( ABC \) is a non degenerate equilateral triangle and \( P \) is the point diametrically opposite to \( A \) in the circumcircle. Prove that \( PA \times PB \times PC = 2R^3 \) where \( R \) is the circumradius.

23. In a triangle, let \( R \) denote the circumradius, \( r \) denote the inradius and \( A \) denote the area. Prove that:
\[
9r^2 \leq A\sqrt{3} \leq r(4R + r)
\]
with equality if, and only if, the triangle is equilateral.

23. If in a triangle, \( O, H, I \) have their usual meanings, prove that
\[
2 \cdot OI \geq IH
\]

24. In acute angled triangle \( ABC \), the circle with diameter \( AB \) intersects the altitude \( CC' \) and its extensions at \( M \) and \( N \) and the circle with diameter \( AC \) intersects the altitude \( BB' \) and its extensions at \( P \) and \( Q \). Prove that \( M, N, P, Q \) are concyclic.

25. Given circles \( C_1 \) and \( C_2 \) which intersect at points \( X \) and \( Y \), let \( \ell_1 \) be a line through the centre of \( C_1 \) which intersects \( C_2 \) at points \( P, Q \). Let \( \ell_2 \) be a line through the centre of \( C_2 \) which intersects \( C_1 \) at points \( R, S \). Show that if \( P, Q, R, S \) lie on a circle then the centre of this circle lies on \( XY \).

26. From a point \( P \) outside a circle, tangents are drawn to the circle, and the points of tangency are \( B, D \). A secant through \( P \) intersects the circle at \( A, C \). Let \( X, Y, Z \) be the feet of the altitudes from \( D \) to \( BC \), \( A \), \( AB \) respectively. Show that \( XY = YZ \).

27. \( \triangle ABC \) is acute and \( h_a, h_b, h_c \) denote its altitudes. \( R, r, r_0 \) denote the radii of its circumcircle, incircle and incircle of its orthic triangle (whose vertices are the feet of its altitudes). Prove the relation:
\[
h_a + h_b + h_c = 2R + 4r + r_0 + \frac{r^2}{R}
\]

28. In a triangle \( \triangle ABC \), points \( D, E, F \) are marked on sides \( BC, CA, AC \), respectively, such that
\[
\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2
\]
Show that
(a) The triangle formed by the lines \( AD, BE, CF \) has an area \( 1/7 \) that of \( \triangle ABC \).
(b) (Generalisation) If the common ratio is \( k \) (greater than 1) then the triangle formed by the lines \( AD, BE, CF \) has an area \( \frac{(k-1)^2}{k^{2}+k+1} \) that of \( \triangle ABC \).
29. Let $AD$, the altitude of $\triangle ABC$ meet the circum-circle at $D'$. Prove that the Simson’s line of $D'$ is parallel to the tangent drawn from $A$.

30. Point $P$ is inside $\triangle ABC$. Determine points $D$ on side $AB$ and $E$ on side $AC$ such that $BD = CE$ and $PD + PE$ is minimum.

31. Prove this result analogous to the Euler Line. In triangle $\triangle ABC$, let $G$, $I$, $N$ be the centroid, incentre, and Nagel point, respectively. Show that,

(a) $I, G, N$ lie on a line in that order, and that $NG = 2 \cdot IG$.

(b) If $P, Q, R$ are the midpoints of $BC, CA, AB$ respectively, then the incentre of $\triangle PQR$ is the midpoint of $TN$.

32. The cyclic quadrilateral $ABCD$ satisfies $AD + BC = AB$. Prove that the internal bisectors of $\angle ADC$ and $\angle BCD$ intersect on $AB$.

33. Let $\ell$ be a line through the orthocentre $H$ of a triangle $\triangle ABC$. Prove that the reflections of $\ell$ across $AB$, $BC$, $CA$ pass through a common point lying on the circumcircle of $\triangle ABC$.

34. If circle $O$ with radius $r_1$ intersect the sides of triangle $ABC$ in six points. Prove that $r_1 \geq r$, where $r$ is the inradius.

35. Construct a right angled triangle given its hypotenuse and the fact that the median falling on hypotenuse is the geometric mean of the legs of the triangle.

36. Find the angles of the triangle which satisfies $R(b + c) = a\sqrt{bc}$ where $a$, $b$, $c$, $R$ are the sides and the circumradius of the triangle.

37. (MOP 1998) Let $ABCDEF$ be a cyclic hexagon with $AB = CD = EF$. Prove that the intersections of $AC$ with $BD$, of $CE$ with $DF$, and of $EA$ with $FB$ form a triangle similar to $\triangle BDF$.

38. $\triangle ABC$ is right-angled and assume that the perpendicular bisectors of $BC$, $CA$, $AB$ cut its incircle ($I$) at three chords. Show that the lengths of these chords form a right-angled triangle.

39. We have a trapezoid $ABCD$ with the bases $AD$ and $BC$. $AD = 4$, $BC = 2$, $AB = 2$. Find possible values of $\angle ACD$.

40. Find all convex polygons such that one angle is greater than the sum of the other angles.

40. If $A_1A_2A_3 \cdots A_n$ is a regular $n$-gon and $P$ is any point on its circumcircle, then prove that

(i) $PA_1^2 + PA_2^2 + PA_3^2 + \cdots + PA_n^2$ is constant;

(ii) $PA_1^4 + PA_2^4 + PA_3^4 + \cdots + PA_n^4$ is constant.
41. In a triangle $ABC$ the incircle $\gamma$ touches the sides $BC$, $CA$, $AD$ at $D$, $E$, $F$ respectively. Let $P$ be any point within $\gamma$ and let the segments $AP$, $BP$, $CP$ meet $\gamma$ at $X$, $Y$, $Z$ respectively. Prove that $DX$, $EY$, $FZ$ are concurrent.

42. $ABCD$ is a convex quadrilateral which has incircle $(I, r)$ and circumcircle $(O, R)$, show that:

$$2R^2 \geq IA \cdot IC + IB \cdot ID \geq 4r^2$$

43. Let $P$ be any point in $\triangle ABC$. Let $AP$, $BP$, $CP$ meet the circumcircle of $\triangle ABC$ again at $A_1$, $B_1$, $C_1$ respectively. $A_2$, $B_2$, $C_2$ are the reflections of $A_1$, $B_1$, $C_1$ about the sides $BC$, $AC$, $AB$ respectively. Prove that the circumcircle of $\triangle A_2B_2C_2$ passes through a fixed point independent of $P$.

44. A point $P$ inside a circle is such that there are three chords of the same length passing through $P$. Prove that $P$ is the center of the circle.

45. $\triangle ABC$ is right-angled with $\angle BAC = 90^\circ$. $H$ is the orthogonal projection of $A$ on $BC$. Let $r_1$ and $r_2$ be the inradii of the triangles $\triangle ABH$ and $\triangle ACH$. Prove

$$AH = r_1 + r_2 + \sqrt{r_1^2 + r_2^2}$$

46. Let $ABC$ be a right angle triangle with $\angle BAC = 90^\circ$. Let $D$ be a point on $BC$ such that the inradius of $\triangle BAD$ is the same as that of $\triangle CAD$. Prove that $AD^2$ is the area of $\triangle ABC$.

47. $\tau$ is an arbitrary tangent to the circumcircle of $\triangle ABC$ and $X$, $Y$, $Z$ are the orthogonal projections of $A$, $B$, $C$ on $\tau$. Prove that with appropriate choice of signs we have:

$$\pm BC\sqrt{AX} \pm CA\sqrt{BY} \pm AB\sqrt{CZ} = 0$$

48. Let $ABCD$ be a convex quadrilateral such that $AB + BC = CD + DA$. Let $I$, $J$ be the incentres of $\triangle BCD$ and $\triangle DAB$ respectively. Prove that $AC$, $BD$, $IJ$ are concurrent.

49. $\triangle ABC$ is equilateral with side length $L$. $P$ is a variable point on its incircle and $A'$, $B'$, $C'$ are the orthogonal projections of $P$ onto $BC$, $CA$, $AB$. Define $\omega_1$, $\omega_2$, $\omega_3$ as the circles tangent to the circumcircle of $\triangle ABC$ at its minor arcs $BC$, $CA$, $AB$ and tangent to $BC$, $CA$, $AB$ at $A'$, $B'$, $C'$ respectively. $\delta_{ij}$ stands for the length of the common external tangent of the circles $\omega_i$, $\omega_j$. Show that $\delta_{12} + \delta_{23} + \delta_{31}$ is constant and compute such value.

50. It is given a triangle $\triangle ABC$ with $AB \neq AC$. Construct a tangent line $\tau$ to its incircle $(I)$ which meets $AC$, $AB$ at $X$, $Y$ such that:

$$\frac{AX}{XC} + \frac{AY}{YB} = 1.$$
51. In $\triangle ABC$, $AB + AC = 3 \cdot BC$. Let the incentre be $I$ and the incircle be tangent to $AB$, $AC$ at $D$, $E$ respectively. Let $D'$, $E'$ be the reflections of $D$, $E$ about $I$. Prove that $BCD'E'$ is cyclic.

52. $\triangle ABC$ has incircle $(I, r)$ and circumcircle $(O, R)$. Prove that, there exists a common tangent line to the circumcircles of $\triangle OBC$, $\triangle OCA$ and $\triangle OAB$ if and only if:

$$\frac{R}{r} = \sqrt{2} + 1$$

53. In a $\triangle ABC$, prove that

$$a \cdot AI^2 + b \cdot BI^2 + c \cdot CI^2 = abc$$

54. In cyclic quadrilateral $ABCD$, $\angle ABC = 90^\circ$ and $AB = BC$. If the area of $ABCD$ is 50, find the length $BD$.

55. Given four points $A, B, C, D$ in a straight line, find a point $O$ in the same straight line such that $OA : OB = OC : OD$.

56. Let the incentre of $\triangle ABC$ be $I$ and the incircle be tangent to $BC$, $AC$ at $E$, $D$. Let $M$, $N$ be midpoints of $AB$, $AC$. Prove that $BI$, $ED$, $MN$ are concurrent.

57. Let $O$ and $H$ be circumcenter and orthocenter of $ABC$ respectively. The perpendicular bisector of $AH$ meets $AB$ and $AC$ at $D$ and $E$ respectively. Show that $\angle AOD = \angle AOE$.

58. Given a semicircle with diameter $AB$ and center $O$ and a line, which intersects the semicircle at $C$ and $D$ and line $AB$ at $M$ ($MB < MA$, $MD < MC$). Let $K$ be the second point of intersection of the circumcircles of $\triangle AOC$ and $\triangle DOB$. Prove that $\angle MKO = 90^\circ$.

59. In the trapezoid $ABCD$, $AB \parallel CD$ and the diagonals intersect at $O$. $P$, $Q$ are points on $AD$ and $BC$ respectively such that $\angle APB = \angle CPD$ and $\angle AQB = \angle CQD$. Show that $OP = OQ$.

60. In cyclic quadrilateral $ABCD$, $\angle ACD = 2\angle BAC$ and $\angle ACB = 2\angle DAC$. Prove that $BC + CD = AC$.

61. $\triangle ABC$ is right with hypotenuse $BC$. $P$ lies on $BC$ and the parallels through $P$ to $AC$, $AB$ meet the circumferences with diameters $PC$, $PB$ again at $U$, $V$ respectively. Ray $AP$ cuts the circumcircle of $\triangle ABC$ at $D$. Show that $\angle UDV = 90^\circ$.

62. Let $ABEF$ and $ACGH$ be squares outside $\triangle ABC$. Let $M$ be the midpoint of $EG$. Show that $\triangle MBC$ is an isoceles right triangle.

63. The three squares $ACC_1A''$, $ABB_1A'$, $BCDE$ are constructed externally on the sides of a triangle $ABC$. Let $P$ be the center of $BCDE$. Prove that the lines $A'C$, $A''B$, $PA$ are concurrent.
64. For triangle \(ABC\), \(AB < AC\), from point \(M\) in \(AC\) such that \(AB + AM = MC\). The straight line perpendicular \(AC\) at \(M\) cut the bisection of \(BC\) in \(I\). Call \(N\) is the midpoint of \(BC\). Prove that is \(MN\) perpendicular to the \(AI\).

65. Let \(ABC\) be a triangle with \(AB \neq AC\). Point \(E\) is such that \(AE = BE\) and \(BE \perp BC\). Point \(F\) is such that \(AF = CF\) and \(CF \perp BC\). Let \(D\) be the point on line \(BC\) such that \(AD\) is tangent to the circumcircle of triangle \(ABC\). Prove that \(D, E, F\) are collinear.

66. Points \(D, E, F\) are outside triangle \(ABC\) such that \(\angle DBC = \angle FBA\), \(\angle DCB = \angle ECA\), \(\angle EAC = \angle FAB\). Prove that \(AD, BE, CF\) are concurrent.

67. In \(\triangle ABC\), \(\angle C = 90^\circ\), and \(D\) is the perpendicular from \(C\) to \(AB\). \(\omega\) is the circumcircle of \(\triangle BCD\). \(\omega_1\) is a circle tangent to \(AC\), \(AB\), and \(\omega\). Let \(M\) be the point of tangency of \(\omega_1\) with \(AB\). Show that \(BM = BC\).

68. Acute triangle \(\triangle ABC\) has orthocenter \(H\) and semiperimeter \(s\). \(r_a, r_b, r_c\) denote its exradii and \(\varrho_a, \varrho_b, \varrho_c\) denote the inradii of triangles \(\triangle HBC\), \(\triangle HCA\) and \(\triangle HAB\). Prove that:

\[ r_a + r_b + r_c + \varrho_a + \varrho_b + \varrho_c = 2s \]

69. The lengths of the altitudes of a triangle are 12,15,20. Find the sides of the triangle and the area of the triangle?

70. Suppose, in an obtuse angled triangle, the orthic triangle is similar to the original triangle. What are the angles of the obtuse triangle?

71. In triangle \(\triangle ABC\) with semiperimeter \(s\), the incircle \((I, r)\) touches side \(BC\) in \(X\). If \(h\) represents the length of the altitude from vertex \(A\) to \(BC\). Show that

\[ AX^2 = 2r \cdot h + (s - a)^2 \]

72. Let \(E, F\) be on \(AB\), \(AD\) of a cyclic quadrilateral \(ABCD\) such that \(AE = CD\) and \(AF = BC\). Prove that \(AC\) bisects the line \(EF\).

73. Suppose \(X\) and \(Y\) are two points on side \(BC\) of triangle \(ABC\) with the following property: \(BX = CY\) and \(\angle BAX = \angle CAY\). Prove \(AB = AC\).

74. \(ABC\) is a triangle in which \(I\) is its incenter. The incircle is drawn and 3 tangents are drawn to the incircle such that they are parallel to the sides of \(ABC\). Now, three triangle are formed near the vertices and their incircles are drawn. Prove that the sum of the radii of the three incircles is equal to the radius of the the incircle of \(ABC\).

75. With usual notation of \(I\), prove that the Euler lines of \(\triangle IBC\), \(\triangle ICA\), \(\triangle IAB\) are concurrent.
76. Vertex $A$ of $\triangle ABC$ is fixed and $B$, $C$ move on two fixed rays $Ax$, $Ay$ such that $AB + AC$ is constant. Prove that the loci of the circumcenter, centroid and orthocenter of $\triangle ABC$ are three parallel lines.

77. $\triangle ABC$ has circumcentre $O$ and incentre $I$. The incentre touches $BC$, $AC$, $AB$ at $D$, $E$, $F$ and the midpoints of the altitudes from $A$, $B$, $C$ are $P$, $Q$, $R$. Prove that $DP$, $EQ$, $FR$, $OI$ are concurrent.

78. The incircle $\Gamma$ of the equilateral triangle $\triangle ABC$ is tangent to $BC$, $CA$, $AB$ at $M$, $N$, $L$. A tangent line to $\Gamma$ through its minor arc $NL$ cut $AB$, $AC$ at $P$, $Q$. Show that:

\[
\frac{1}{[MPB]} + \frac{1}{[MQC]} = \frac{6}{[ABC]}
\]

79. $A$ and $B$ are on a circle with center $O$ such that $AOB$ is a quarter of the circle. Square $OEDC$ is inscribed in the quarter circle, with $E$ on $OB$, $D$ on the circle, and $C$ on $OA$. Let $F$ be on arc $AD$ such that $CD$ bisects $\angle FCB$. Show that $BC = 3 \cdot CF$.

80. Take a circle with a chord drawn in it, and consider any circle tangent to both the chord and the minor arc. Let the point of tangency for the small circle and the chord be $X$. Also, let the point of tangency for the small circle and the minor arc be $Y$. Prove that all lines $XY$ are concurrent.

81. Two circles intersect each other at $A$ and $B$. Line $PT$ is a common tangent, where $P$ and $T$ are the points of tangency. Let $S$ be the intersection of the two tangents to the circumcircle of $\triangle APT$ at $P$ and $T$. Let $H$ be the reflection of $B$ over $PT$. Show that $A$, $H$, and $S$ are collinear.

82. In convex hexagon $ABCDEF$, $AD = BC + EF$, $BE = CD + AF$ and $CF = AB + DE$. Prove that

\[
\frac{AB}{DE} = \frac{CD}{AF} = \frac{EF}{BC}.
\]

83. The triangle $ABC$ is scalene with $AB > AC$. $M$ is the midpoint of $BC$ and the angle bisector of $\angle BAC$ hits the segment $BC$ at $D$. $N$ is the perpendicular foot from $C$ to $AD$. Given that $MN = 4$ and $DM = 2$. Compute the value $AM^2 - AD^2$.

84. $A$, $B$, $C$, and $D$ are four points on a line, in that order. Isosceles triangles $AEB$, $BFC$, and $CGD$ are constructed on the same side of the line, with $AE = EB = BF = FC = CG = GD$. $H$ and $I$ are points so that $BEHF$ and $CFIG$ are rhombi. Finally, $J$ is a point such that $FHJI$ is a rhombus. Show that $JA = JD$.

85. A line through the circumcenter $O$ of $\triangle ABC$ meets sides $AB$ and $AC$ at $M$ and $N$, respectively. Let $R$ and $S$ be the midpoints of $CM$ and $BN$ respectively. Show that $\angle BAC = \angle ROS$. 

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86. Let $AB$ be a chord in a circle and $P$ a point on the circle. Let $Q$ be the foot of the perpendicular from $P$ to $AB$, and $R$ and $S$ the feet of the perpendiculars from $P$ to the tangents to the circle at $A$ and $B$. Prove that $PQ^2 = PR \cdot PS$.

87. Given a circle $\omega$ with diameter $AB$, a line outside the circle $d$ is perpendicular to $AB$ closer to $B$ than $A$. $C \in \omega$ and $D = AC \cap d$. A tangent from $D$ is drawn to $E$ on $\omega$ such that $B, E$ lie on same side of $AC$. $F = BE \cap d$ and $G = FA \cap \omega$ and $G' = FC \cap \omega$. Show that the reflection of $G$ across $AB$ is $G'$.

88. $\triangle ABC$ is acute and its angles $\alpha, \beta, \gamma$ are measured in radians. $S$ and $S_0$ represent the area of $\triangle ABC$ and the area bounded/overlapped by the three circles with diameters $BC, CA, AB$ respectively. Show that:

$$S + 2S_0 = \frac{a^2}{2} \left( \frac{\pi}{2} - \alpha \right) + \frac{b^2}{2} \left( \frac{\pi}{2} - \beta \right) + \frac{c^2}{2} \left( \frac{\pi}{2} - \gamma \right)$$

89. $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle A = 30^\circ$. The triangle is inscribed in a circle with center $O$. The point $D$ lies on the arch between $A$ and $C$ such that $\angle DOC = 30^\circ$. Let $G$ be the point on the arch between $A$ and $B$ such that $AC = DG$ and $AG < BG$. The line $DG$ intersects $AC$ and $AB$ in $E$ and $F$ respectively.

(a) Prove that $\triangle AFG$ is equilateral.

(b) Find the ratio between the areas $\frac{\triangle AFG}{\triangle ABC}$.

90. Construct a triangle $ABC$ given the lengths of the altitude, median and inner angle bisector emerging from vertex $A$.

91. Let $P$ be a point in $\triangle ABC$ such that $\frac{AB}{PC} = \frac{AP}{PC}$. Prove that $\angle PBC + \angle PAC = \angle PBA + \angle PCA$.

92. Point $D$ lies inside the equilateral $\triangle ABC$, such that $DA^2 = DB^2 + DC^2$. Show that $\angle BDC = 150^\circ$.

93. (China MO 1998) Find the locus of all points $D$ with respect to a given triangle $\triangle ABC$ such that

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

94. Let $P$ be a point in equilateral triangle $ABC$. If $\angle BPC = \alpha$, $\angle CPA = \beta$, $\angle APB = \gamma$, find the angles of the triangle with side lengths $PA, PB, PC$.

95. Of a $\square ABCD$, let $P, Q, R, S$ be the midpoints of the sides $AB, BC, CD, DA$. Show that if $\triangle AQR$ and $\triangle CSP$ are equilateral, then $\square ABCD$ is a rhombus. Also find its angles.
96. In \( \triangle ABC \), the incircle touches \( BC \) at the point \( X \). \( A' \) is the midpoint of \( BC \). \( I \) is the incentre of \( \triangle ABC \). Prove that \( A'I \) bisects \( AX \).

97. In convex quadrilateral \( ABCD \), \( \angle BAC = 80^\circ \), \( \angle BCA = 60^\circ \), \( \angle DCA = 70^\circ \), \( \angle DCA = 40^\circ \). Find \( \angle DBC \).

98. It is given a \( \triangle ABC \), and let \( X \) be an arbitrary point inside the triangle. If \( XD \perp AB \), \(XE \perp BC \), \( XF \perp AC \), where \( D \in AB \), \( E \in BC \), \( F \in AC \), then prove that:

\[
AX + BX + CX \geq 2(XD +XE +XF)
\]

99. Let \( A_1 \), \( A_2 \), \( A_3 \) and \( A_4 \) be four circles such that the circles \( A_1 \) and \( A_3 \) are tangential at a point \( P \), and the circles \( A_2 \) and \( A_4 \) are also tangential at the same point \( P \). Suppose that the circles \( A_1 \) and \( A_2 \) meet at a point \( T_1 \), the circles \( A_2 \) and \( A_3 \) meet at a point \( T_2 \), the circles \( A_3 \) and \( A_4 \) meet at a point \( T_3 \), and the circles \( A_4 \) and \( A_1 \) meet at a point \( T_4 \), such that all these four points \( T_1 \), \( T_2 \), \( T_3 \), \( T_4 \) are distinct from \( P \). Prove that

\[
\left( \frac{T_1 T_2}{T_2 T_3} \right) \cdot \left( \frac{T_2 T_3}{T_3 T_4} \right) = \left( \frac{PT_2}{PT_4} \right)^2
\]

100. \( ABCD \) is a convex quadrilateral such that \( \angle ADB + \angle ACB = 180^\circ \). It’s diagonals \( AC \) and \( BD \) intersect at \( M \). Show that

\[
AB^2 = AM \cdot AC + BM \cdot BD
\]

101. Let \( AH \), \( BM \) be the altitude and median of triangle \( ABC \) from \( A \) and \( B \). If \( AH = BM \), find \( \angle MBC \).

102. \( P \), \( Q \), \( R \) are random points in the interior of \( BC \), \( CA \), and \( AB \) respectively of a non-degenerate triangle \( ABC \) such that the circumcircles of \( BPR \) and \( CQP \) are orthogonal and intersect in \( M \) other than \( P \). Prove that \( PR \cdot MQ \), \( PQ \cdot MR \), \( QR \cdot MP \) can be the sides of a right angled triangle.

103. \( \triangle ABC \) is scalene and \( D \) is a point on the arc \( BC \) of its circumscribed circle which doesn’t contain \( A \). Perpendicular bisectors of \( AC \), \( AB \) cut \( AD \) at \( Q \), \( R \). If \( P = BR \cap CQ \), then show that \( AD = PB + PC \).

104. It is given \( \triangle ABC \) and \( M \) is the midpoint of the segment \( AB \). Let \( \ell \) pass through \( M \) and \( \ell \cap AC = K \) and \( \ell \cap BC = L \), such that \( CK = CL \). Let \( CD \perp AB \), \( D \in AB \) and \( O \) is the center of the circle, circumscribed around \( \triangle CKL \). Prove that \( OM = OD \).

105. Prove that: The locus of points \( P \) in the plane of an acute triangle \( \triangle ABC \) which satisfy that the length of segments \( PA \), \( PB \), \( PC \) can form a right triangle is the union of three circumferences, whose centers are the reflections of \( A \), \( B \), \( C \) across the midpoints of \( BC \), \( CA \), \( AB \) and whose radii are given by \( \sqrt{b^2 + c^2 - a^2} \), \( \sqrt{a^2 + c^2 - b^2} \), \( \sqrt{a^2 + b^2 - c^2} \).
106. Let $D, E$ be points on the rays $BA, CA$ respectively such that $BA \cdot BD + CA \cdot CE = BC^2$. Prove that $\angle CDA = \angle BEC$.

107. In triangle $ABC$, $M, N, P$ are points on sides $BC, CA, AB$ respectively such that perimeter of the triangle $MNP$ is minimal. Prove that triangle $MNP$ is the orthic triangle of $ABC$ (the triangle formed by the foot of the perpendiculars on the sides as vertices).

108. Prove that there exists an inversion mapping two non-intersecting circles into concentric circles.

109. Let $\alpha, \beta, \gamma$ be three circles concurring at $M$. $AM, BM, CM$ are the common chords of $\alpha, \beta, \gamma$; $\alpha, \beta, \gamma$; and $\gamma, \alpha, \beta$ respectively. $AM, BM, CM$ intersect $\gamma, \alpha, \beta$ at $P, Q, R$ respectively. Prove that $AQ \cdot BR \cdot CP = AR \cdot BP \cdot CQ$.

110. In triangle $\triangle ABC$, lines $\ell_b$ and $\ell_c$ are perpendicular to $BC$ through vertices $B, C$ respectively. $P$ is a variable point on line $BC$ and the perpendicular lines dropped from $P$ to $AB, AC$ cut $\ell_b, \ell_c$ at $U, V$ respectively. Show that $UV$ always passes through the orthocenter of $\triangle ABC$.

111. Let $I$ be the incenter of triangle $ABC$ and $M$ is the midpoint of $BC$. The excircle opposite $A$ touches the side $BC$ at $D$. Prove that $AD \parallel IM$.

112. An incircle of $ABC$ triangle tangents $BC, CA$ and $AB$ sides at $A_1, B_1$ and $C_1$ points, respectively. Let $O$ and $I$ be circumcenter and incenter and $OI \cap BC = D$. A line through $A_1$ point and perpendicular to $B_1C_1$ cut $AD$ at $E$. Prove that $M$ point lies on $B_1C_1$ line. ($M$ is midpoint of $EA_1$).

113. Parallels are drawn to the sides of the triangle $ABC$ such that the lines touch the in-circle of $ABC$. The lengths of the tangents within $ABC$ are $x, y, z$ respectively opposite to sides $a, b, c$ respectively. Prove the relation: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

114. In an acute angled triangle $ABC$, the points $D, E, F$ are on sides $BC, CA, AB$ respectively, such that $\angle AFE = \angle BFD, \angle FDB = \angle EDC, \angle DEC = \angle FEA$. Prove that $DEF$ is the orthic triangle of $ABC$.

115. Let $\omega$ be circle and tangents $AB, AC$ sides and circumcircle/externally and at $D$ point. Prove that circumcenter of $\triangle ABC$ lies on bisector of $\angle BDC$.

116. Construct a triangle with ruler-compass operations, given its inradius, circumradius and any altitude.

117. Let $AD, BM, CH$ be the angle bisector, median, altitude from $A, B, C$ of $\triangle ABC$. If $AD = BM = CH$, prove that $\triangle ABC$ is equilateral.
118. Consider a triangle $ABC$ with $BC = a$, $CA = b$, $AB = c$ and area equal to 4. Let $x$, $y$, $z$ the distances from the orthocenter to the vertices $A$, $B$, $C$. Prove that if $a\sqrt{x} + b\sqrt{y} + c\sqrt{z} = 4\sqrt{a+b+c}$, then $ABC$ is equilateral.

119. Suppose that $\angle A$ is the smallest of the three angles of triangle $ABC$. Let $D$ be a point on the arc $BC$ of the circumcircle of $\triangle ABC$ not containing $A$. Let the perpendicular bisectors of $AB$, $AC$ intersect $AD$ at $M$ and $N$ respectively. Let $BM$ and $CN$ meet at $T$. Prove that $BT + CT \leq 2R$ where $R$ is the circumradius of triangle $ABC$.

120. Points $E$, $F$ are taken on the side $AB$ of triangle $ABC$ such that the lengths of $CE$ and $CF$ are both equal to the semiperimeter of the triangle $ABC$. Prove that the circumcircle of $CEF$ is tangent to the excircle of triangle $ABC$ opposite $C$.

121. Two fixed circles $\omega_1$, $\omega_2$ intersect at $A$, $B$. A line $\ell$ through $A$ cuts $\omega_1$, $\omega_2$ again at $U$, $V$. Show that the perpendicular bisector of $UV$ goes through a fixed point as line $\ell$ spins around $A$.

122. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let $X$ and $Y$ be points on sides $BC$ and $CA$ such that $XY \parallel AB$. Let $D$ be the circumcenter of $\triangle CXY$ and $E$ be the midpoint of $BY$. Prove that $\angle AED = 90^\circ$.

123. Tetrahedron $ABCD$ is featured on ball (centre $S$, $r = 1$) and $SA \geq SB \geq SD$. Prove that $SA > \sqrt{5}$.

124. Let $ABCD$ be a cyclic quadrilateral. The lines $AB$ and $CD$ intersect at the point $E$, and the diagonals $AC$ and $BD$ at the point $F$. The circumcircle of the triangles $AFD$ and $BFC$ intersect again at $H$. Prove that $EHF = 90^\circ$.

125. $ABCD$ is a cyclic and circumscribed quadrilateral whose incircle touches the sides $AB$, $BC$, $CD$, $DA$ at $E$, $F$, $G$, $H$. Prove that $EG \perp FH$.

126. Let $\tau$ be an arbitrary tangent line to the circumcircle $(O, R)$ of $\triangle ABC$. $\delta(P)$ stands for the distance from point $P$ to $\tau$. If $I$, $I_a$, $I_b$, $I_c$ denote the incenter and the three excenters of $\triangle ABC$, prove with appropriate choice of signs that:

$$\pm \delta(I) \pm \delta(I_a) \pm \delta(I_b) \pm \delta(I_c) = 4R$$

127. Let $ABC$ be a fixed triangle and $\beta$, $\gamma$ are fixed angles. Let $\alpha$ be a variable angle. Let $E$, $F$ be points outside $\triangle ABC$ such that $\angle FBA = \beta$, $\angle FAB = \alpha$, $\angle ECA = \gamma$, $\angle EAC = \alpha$. Prove that the intersection of $BE$, $CF$ lies on a fixed line independent of $\alpha$.

128. Incircle $(I)$ of $\triangle ABC$ touches $BC$, $CA$, $AB$ at $D$, $E$, $F$ and $BI$, $CI$ cut $CA$, $AB$ at $M$, $N$. Line $MN$ intersects $(I)$ at two points, let $P$ be one of these points. Show that the lengths of segments $PD$, $PE$, $PF$ form a right triangle.

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129. Given a triangle \( ABC \) with orthocentre \( H \), circumcentre \( O \), incentre \( I \) and \( D \) is the tangency point of incircle with \( BC \). Prove that if \( OI \) and \( BC \) are parallel, then \( AO \) and \( HD \) are parallel as well.

130. Let \( ABCD \) be a cyclic quadrilateral such that \( \frac{AB}{BD} = \frac{AD}{DC} \). The circle passing through \( A, B \) and tangent to \( AD \) intersects \( CB \) at \( E \). The circle passing through \( A, D \) and tangent to \( AB \) intersects \( CD \) at \( F \). Prove that \( BEFD \) is cyclic.

131. Given two points \( A, B \) and a circle \( (O) \) not containing \( A, B \). Consider the radical axis of an arbitrary circle passing through \( A, B \) and \( (O) \). Prove that all such radical axes passes through a fixed point \( P \) and construct it.

132. Given a sphere of radius one that tangents the six edges of an arbitrary tetrahedron. Find the maximum possible volume of the tetrahedron.

133. Let \( ABC \) be a triangle for which exists \( D \in BC \) so that \( AD \perp BC \). Denote \( r_1, r_2 \) the lengths of inradius for the triangles \( ABD, ADC \) respectively. Prove that

\[
ar_1 + (s - a)(s - c) = ar_2 + (s - a)(s - b) = sr
\]

134. Let \( M \) be the midpoint of \( BC \) of triangle \( ABC \). Suppose \( D \) is a point on \( AM \). Prove that \( \angle DBC = \angle DAB \) if and only if \( \angle DCB = \angle DAC \).

135. \( P \) and \( R \) are two given points on a circle \( \Omega \). Let \( O \) be an arbitrary point on the perpendicular bisector of \( PR \). A circle with centre \( O \) intersects \( OP \) and \( OR \) at the points \( M, N \) respectively. The tangents to this circle at \( M \) and \( N \) meet \( \omega \) at points \( Q \) and \( S \) respectively such that \( P, Q, R, S \) lie on \( \Omega \) in this order. \( PQ \) and \( RS \) intersect at \( K \). Show that the line joining the midpoints of \( PQ \) and \( RS \) is perpendicular to \( OK \).

136. In cyclic quadrilateral \( ABCD \), \( AB = 8 \), \( BC = 6 \), \( CD = 5 \), \( DA = 12 \). Let \( AB \) intersect \( DC \) at \( E \). Find the length \( EB \).

137. In triangle \( ABC \), let \( \Gamma \) be a circle passing through \( B \) and \( C \) and intersecting \( AB \) and \( AC \) at \( M, N \) respectively. Prove that the locus of the midpoint of \( MN \) is the \( A \)-symmedian of the triangle.

138. Let \( ABC \) be a triangle \( E \) is the excenter of \( \triangle ABC \) opposite \( A \). If \( AC + CB = AB + BE \), find \( \angle ABC \).

139. In a given line segment \( AB \), choose an arbitrary point \( C \) in the interior. The point \( D, E, \ F \) are the midpoints of the segments \( AC, CB \) and \( AB \) respectively, and consider the point \( X \) in the interior of the line segment \( CF \) such that \( \frac{BX}{DX} = 2 \). Prove that

\[
\frac{BX}{DX} = \frac{AX}{XE} = 2
\]
140. Diagonals of a convex quadrilateral with an area of $Q$ divide it into four triangles with appropriate areas $P_1$, $P_2$, $P_3$, $P_4$. Prove that

$$P_1 \cdot P_2 \cdot P_3 \cdot P_4 = \frac{(P_1 + P_2)^2 \cdot (P_2 + P_3)^2 \cdot (P_3 + P_4)^2 \cdot (P_4 + P_1)^2}{Q^4}$$

141. Let the incircle $\omega$ of a triangle $\triangle ABC$ touches its sides $BC$, $CA$, $AB$ at the points $D$, $E$, $F$ respectively. Now, let the line parallel to $AB$ through $E$ meets $DF$ at $Q$, and the parallel to $AB$ through $D$ meets $EF$ at $T$. Prove that the lines $CF$, $DE$, $QT$ are concurrent.

142. $ABCDEF$ is a hexagon whose opposite sides are parallel, this is, $AB \parallel DE$, $BC \parallel EF$ and $CD \parallel FA$. Show that triangles $\triangle ACE$ and $\triangle BDF$ have equal area.

143. Given a circle $\omega$ and a point $A$ outside it. Construct a circle $\gamma$ with centre $A$ orthogonal to $\omega$.

144. Prove that the circumcircles of the four triangles in a complete quadrilateral meet at a point. (Miquel Point)

145. Prove that the symmedian point of a triangle is the centroid of it’s pedal triangle with respect to that triangle.

146. Quadrilateral $ABCD$ is convex with circumcircle $(O)$, $O$ lies inside $ABCD$. Its diagonals $AC$, $BD$ intersect at $S$ and let $M$, $N$, $L$, $P$ be the orthogonal projections of $S$ onto sides $AB$, $BC$, $CD$, $DA$. Prove that $[ABCD] \geq 2[MNLP]$.

147. Let $\omega$ be a circle in which $AB$ and $CD$ are parallel chords and $\ell$ is a line from $C$, that intersects $AB$ in its midpoint $L$ and $\ell \cap \omega = E$. $K$ is the midpoint of $DE$. Prove that $KE$ is the angle bisector of $\angle AKB$.

148. Let $ABC$ be an equilateral triangle and $D$, $E$ be on the same side as $C$ with the line $AB$, and $BD$ is between $BA$, $BE$. Suppose $\angle DBE = 90^\circ$, $\angle EDB = 60^\circ$. Let $F$ be the reflection of $E$ about the point $C$. Prove that $FA \perp AD$.

149. In cyclic quadrilateral $ABCD$, $AC \cdot BD = 2 \cdot AB \cdot CD$. $E$ is the midpoint of $AC$. Prove that circumcircle of $ADE$ is tangential to $AB$.

150. $ABCD$ is a rhombus with $\angle BAD = 60^\circ$. Arbitrary line $\ell$ through $C$ cuts the extension of its sides $AB$, $AD$ at $M$, $N$ respectively. Prove that lines $DM$ and $BN$ meet on the circumcircle of $\triangle BAD$.

151. Let $ABC$ be a triangle. Prove that there is a line(in the plane of $ABC$) such that the intersection of the interior of triangle $ABC$ and interior of its reflection $A'B'C'$ has more than $2/3$ the area of triangle $ABC$. 

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152. In triangle $ABC$, $D$, $E$, $F$ are feet of perpendiculars from $A$, $B$, $C$ to $BC$, $AC$, $AB$. Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle DEF$.

153. Let $ABC$ be a triangle right-angled at $A$ and $\omega$ be its circumcircle. Let $\omega_1$ be the circle touching the lines $AB$ and $AC$, and the circle $\omega$ internally. Further, let $\omega_2$ be the circle touching the lines $AB$ and $AC$ and the circle $\omega$ externally. If $r_1$, $r_2$ be the radii of $\omega_1$, $\omega_2$ prove that $r_1 \cdot r_2 = 4A$ where $A$ is the area of the triangle $ABC$.

154. The points $D$, $E$ and $F$ are chosen on the sides $BC$, $AC$ and $AB$ of triangle $ABC$, respectively. Prove that triangles $ABC$ and $DEF$ have the same centroid if and only if

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}$$

155. Tangents to a circle form an external point $A$ are drawn meeting the circle at $B$, $C$ respectively. A line passing through $A$ meets the circle at $D$, $E$ respectively. $F$ is a point on the circle such that $BF$ is parallel to $DE$. Prove that $FC$ bisects $DE$.

156. Let $E$ be the intersection of the diagonals of the convex quadrilateral $ABCD$. Define $[T]$ to be the area of triangle $T$. If $[ABE] + [CDE] = [BCE] + [DAE]$, prove that one of the diagonals bisect the other.

157. A line intersects $AB$, $BC$, $CD$, $DA$ of quadrilateral $ABCD$ in the points $K$, $L$, $M$, $N$. Prove that

$$\frac{AK}{KB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MD} \cdot \frac{DN}{NA} = 1$$

in magnitudes.

158. Let $PQ$ be a chord of a circle. Let the midpoint of $PQ$ be $M$. Let $AB$ and $CD$ be two chords passing through $M$. Let $AC$ and $BD$ meet $PQ$ at $H$, $K$ respectively. Prove that

$$\frac{HAHC}{HM^2} = \frac{KBKD}{KM^2}$$

159. Let $ABCD$ be a trapezium with $AB \parallel CD$. Prove that

$$(AB^2+AC^2-BC^2)(DB^2+DC^2-BC^2) = (BA^2+BD^2-AD^2)(CA^2+CD^2-AD^2)$$

160. Given a rectangle $ABCD$ and a point $P$ on its boundary. Let $S$ be the sum of the distances of $P$ from $AC$ and $BD$. Prove that $S$ is constant as $P$ varies on the boundary.

161. Let $P$ and $Q$ be two points on a semicircle whose diameter is $XY$ ($P$ nearer to $X$). Join $XP$ and $YQ$ and let them meet at $B$. Let the tangents from $P$ and $Q$ meet at $R$. Prove that $BR$ is perpendicular to $XY$. 

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162. Let a cyclic quadrilateral $ABCD$. $L$ is the intersection of $AC$ and $BD$ and $S = AD \cap BC$. Let $M$, $N$ is midpoints of $AB$, $CD$. Prove that $SL$ is a tangent of $(MNL)$. 

163. Let $ABC$ be a right triangle with $\angle A = 90^\circ$. Let $D$ be such that $CD \perp BC$. Let $O$ be the midpoint of $BC$. $DO$ intersect $AB$ at $E$. Prove that $\angle ECB = \angle ADC + \angle ACD$.

164. Given a circle $\omega$ and a point $A$ outside it. A circle $\omega'$ passing through $A$ is tangential to $\omega$ at $B$. The tangents to $\omega'$ at $A$, $B$ intersect in $M$. Find the locus of $M$.

165. Triangle $\triangle ABC$ has incircle $(I)$ and circumcircle $(O)$. The circle with center $A$ and radius $AI$ cuts $(O)$ at $X$, $Y$. Show that line $XY$ is tangent to $(I)$.

166. Let $ABCD$ be a cyclic quadrilateral with circumcircle $\omega$. Let $AB$ intersect $DC$ at $E$. The tangent to $\omega$ at $D$ intersect $BC$ at $F$. The tangent to $\omega$ at $C$ intersect $AD$ at $G$. Prove that $E$, $F$, $G$ are collinear.

167. Let $ABC$ is a right triangle with $C = 90^\circ$. $H$ is the leg of the altitude from $C$, $M$ is the mid-point of $AB$. $P$ is a point in $ABC$ such that $AP = AC$. Prove that $PM$ is the bisector of $\angle HPA$ if and only if $A = 60^\circ$.

168. Two circles $w_1$ and $w_2$ meets at points $P,Q$. $C$ is any point on $w_1$ different from $P,Q$. $CP$ meets $w_2$ at point $A$. $CQ$ meets $w_2$ at point $B$. Find locus for $ABC$ triangle’s circumcircle’s centres.

169. Consider a triangle $\triangle ABC$ with incircle $(I)$ touching its sides $BC$, $CA$, $AB$ at $A_0$, $B_0$, $C_0$ respectively. The triangle $\triangle A_0B_0C_0$ is called the intouch triangle of $\triangle ABC$. Likewise, the triangle formed by the points of tangency of an excircle with the sidelines of $\triangle ABC$ is called an extouch triangle. Let $S_0$, $S_1$, $S_2$, $S_3$ denote the areas of the intouch triangle and the three extouch triangles respectively. Show that:

$$\frac{1}{S_0} = \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3}$$

170. Let $ABCD$ be a convex quadrilateral such that $\angle DAB = 90^\circ$ and $DA = DC$. Let $E$ be on $CD$ such that $EA \perp BD$. Let $F$ be on $BD$ such that $FC \perp DC$. Prove that $BC \parallel FE$.

171. (China TST 2007) Let $\omega$ be a circle with centre $O$. Let $A$, $B$ be two points on its perimeter, and let $CS$ and $CT$ be two tangents drawn to $\omega$ from a point $C$ outside the circle. Let $M$ be the midpoint of the minor arc $\overline{AB}$. $MS$ and $MT$ intersect $\overline{AB}$ in $E$, $F$ respectively. The lines passing through $E$, $F$ perpendicular to $AB$ cut $OS$, $OT$ at $X$ and $Y$ respectively. Let $\ell$ be an arbitrary line cutting $\omega$ at the points $P$ and $Q$ respectively. Denote $R = MP \cap AB$. If $Z$ is the circumcentre of triangle $PQR$, prove that $X$, $Y$, $Z$ are collinear.
172. Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \angle ADC$. Let $E$ be the foot of perpendicular from $A$ to $BC$ and $F$ is the foot of perpendicular from $A$ to $CD$. Let $M$ be the midpoint of $BD$. Prove that $ME = MF$.

173. Let $H, K, I$ be the feet of the altitude from $A, B, C$ of triangle $ABC$. Let $M, N$ be the feet of the altitude from $K, I$ of triangle $AIK$. Let $P, Q$ be the point on $HI, HK$ such that $AP, AQ$ be perpendicular to $HI, HK$ respectively. Prove that $M, N, P, Q$ are collinear.

174. We have a $\triangle ABC$ with $\angle BAC = 90^\circ$. $D$ is constructed such that $AB = BD$ and $A, B, D$ are three different collinear points. $X$ is the foot of the altitude through $A$ in $\triangle ABC$. $Y$ is the midpoint of $CX$. Construct the circle $\tau$ with diameter $CX$. $AC$ intersects $\tau$ again in $F$ and $AY$ intersects $\tau$ at $G, H$. Prove that $DX, CG, HF$ are concurrent.

175. Let $ABCDE$ be a convex pentagon such that $\angle EAB = 90^\circ$, $EB = ED$, $AB = DC$ and $AB \parallel DC$. Prove that $\angle BED = 2\angle CAB$.

176. A straight line intersects the $AB, BC$ internally and $AC$ externally of triangles $ABC$ in the points $D, E, F$ respectively. Prove that the midpoints of $AE, BF, CD$ are collinear.

177. Inside an acute triangle $ABC$ is chosen point point $K$, such that $\angle AKC = 2\angle ABC$ and $\frac{AK}{KC} = \left(\frac{AB}{BC}\right)^2$. where $A_1$ and $C_1$ are the midpoints of $BC$ and $AB$. Prove, that $K$ lies on circumcircle of triangle $A_1BC_1$.

178. $M$ is the midpoint of the side $BC$ of $\triangle ABC$ and $AC = AM + AB$. Incircle $(I)$ of $\triangle ABC$ cuts $A$-median $AM$ at $X, Y$. Show that $\angle XAY = 120^\circ$.

179. Let $ABC$ be an isosceles triangle with $AB = AC$. Let $P, Q$ be points on the side $BC$ such that $\angle APC = 2\angle AQB$. Prove that $BP = AP + QC$.

180. Let $BC$ be a diameter of the circle $O$ and let $A$ be an interior point. Suppose that $BA$ and $CA$ intersect the circle $O$ at $D$ and $E$, respectively. If the tangents to the circle $O$ at $E$ and $D$ intersect at the point $M$, prove that $AM$ is perpendicular to $BC$.

181. Let $ABC$ be triangle and $G$ its centroid. Then for any point $M$, we have

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2.$$ 

182. Given two non-intersecting and non-overlapping circles and a point $A$ lying outside the circles. Prove that there are exactly four circles(straight lines are also considered as circles) touching the given two circles and passing through $A$. 

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183. A non-isosceles triangle $ABC$ is given. The altitude from $B$ meets $AC$ at $E$. The line through $E$ perpendicular to the $B$-median meets $AB$ at $F$ and $BC$ at $G$. Prove that $EF = EG$ if, and only if, $\angle ABC = 90^\circ$.

184. Given a triangle $ABC$ and a point $T$ on the plane whose projections on $AB$, $AC$ are $C_1$, $B_1$ respectively. $B_2$ is on $BT$ such that $AB_2$ is perpendicular to $BT$ and $C_2$ is on $CT$ such that $AC_2$ is perpendicular to $CT$. Prove that $B_1B_2$ and $C_1C_2$ intersect on $BC$.

185. Let $ABCD$ be a cyclic quadrilateral with $\angle BAD = 60^\circ$. Suppose $BA = BC + CD$. Prove that either $\angle ABD = \angle CBD$ or $\angle ABC = 60^\circ$.

186. In a quadrilateral $ABCD$ we have $AB \parallel CD$ and $AB = 2 \cdot CD$. A line $\ell$ is perpendicular to $CD$ and contains the point $C$. The circle with centre $D$ and radius $DA$ intersects the line $\ell$ at points $P$ and $Q$. Prove that $AP \perp BQ$.

187. In triangle $ABC$, a circle passes through $A$ and $B$ and is tangent to $BC$. Also, a circle that passes through $B$ and $C$ is tangent to $AB$. These two circles intersect at a point $K$ other than $B$. If $O$ is the circumcenter of $ABC$, prove that $\angle BKO = 90^\circ$.

188. Four points $P, Q, R, S$ are taken on the sides $AB$, $BC$, $CD$, $DA$ of a quadrilateral such that
\[
\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1
\]
Prove that $PQ$ and $RS$ intersect on $AC$.

189. Let $D$ be the midpoint of $BC$ of triangle $ABC$. Let its incenter be $I$ and $AI$ intersects $BC$ at $E$. Let the excircle opposite $A$ touches the side $BC$ at $F$. Let $M$ be the midpoint of $AF$. Prove that $AD, FI, EM$ are concurrent.

190. $\triangle ABC$ is scalene and its $B$- and $C$- excircles $(I_b)$ and $(I_c)$ are tangent to sideline $BC$ at $U$, $V$. $M$ is the midpoint of $BC$ and $P$ is its orthogonal projection onto line $I_bI_c$. Prove that $A, U, V, P$ are concyclic.

191. Let $H$ be the orthocenter of acute $\triangle ABC$. Let $D$, $E$, $F$ be feet of perpendiculars from $A$, $B$, $C$ onto $BC$, $CA$, $AB$ respectively. Suppose the squares constructed outside the triangle on the sides $BC$, $CA$, $AB$ has area $S_a$, $S_b$, $S_c$ respectively. Prove that
\[
S_a + S_b + S_c = 2(AH \cdot AD + BH \cdot BE + CH \cdot CF)
\]

192. In rectangle $ABCD$, $E$ is the midpoint of $BC$ and $F$ is the midpoint of $AD$. $G$ is a point on $AB$ (extended if necessary); $GF$ and $BD$ meet at $H$. Prove that $EF$ is the bisector of angle $GEH$. 

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193. \( P \) is a point in the minor arc \( BC \) of the circumcircle of a square \( ABCD \), prove that
\[
\frac{PA + PC}{PB + PD} = \frac{PD}{PA}
\]

194. \( ABCD \) is a cyclic trapezoid with \( AB \parallel CD \). \( M \) is the midpoint of \( CD \) and \( AM \) cuts the circumcircle of \( ABCD \) again at \( E \). \( N \) is the midpoint of \( BE \). Show that \( NE \) bisects \( \angle CND \).

195. A line is drawn passing though the centroid of a \( \triangle ABC \) meeting \( AB \) and \( AC \) at \( M \) and \( N \) respectively. Prove that
\[
AM \cdot NC + AN \cdot MB = AM \cdot AN
\]

196. Let the isosceles triangle \( ABC \) where \( AB = AC \). The point \( D \) belongs to the side \( BC \) and the point \( E \) belongs to \( AC \). \( C = 50^\circ \), \( \angle ABD = 80^\circ \) and \( \angle ABE = 30^\circ \); find \( \angle BDE \).

197. Let \( S \) be the area of \( \triangle ABC \) and \( BC = a \). Let \( r \) be its inradius and \( r_a \) be its exradius opposite \( A \). Prove that
\[
S = \frac{arr_a}{r_a - r}
\]

198. A line segment \( AB \) is divided by internal points \( K, L \) such that \( AL^2 = AK \cdot AB \). A circle with centre \( A \) and radius \( AL \) is drawn. For any point \( P \) on the circle, prove that \( PL \) bisects \( \angle KPB \).

199. Let \( \triangle ABC \) be a triangle with \( \angle A = 60^\circ \). Let \( BE \) and \( CF \) be the internal angle bisectors of \( \angle B \) and \( \angle C \) with \( E \) on \( AC \) and \( F \) on \( AB \). Let \( M \) be the reflection of \( A \) in the line \( EF \). Prove that \( M \) lies on \( BC \). (Regional Olympiad 2010, India)

200. In triangle \( ABC \), \( Z \) is a point on the base \( BC \). Lines passing though \( B \) and \( C \) that are parallel to \( AZ \) meet \( AC \) and \( AB \) at \( X, Y \) respectively. Prove that:
\[
\frac{1}{BX} + \frac{1}{CY} = \frac{1}{AZ}
\]

201. Let \( ABCD \) be a trapezoid such that \( AB > CD \), \( AB \parallel CD \). Points \( K \) and \( L \) lie on the segments \( AB \) and \( CD \) respectively such that \( \frac{AK}{KB} = \frac{DL}{LC} \). Suppose that there are points \( P \) and \( Q \) on the segment \( KL \) satisfying \( \angle APB = \angle BCD \) and \( \angle CQD = \angle ABC \). Prove that \( P, Q, B, C \) are concyclic.

202. \( I \) is the incenter of \( \triangle ABC \). Let \( E \) be on the extension of \( CA \) such that \( CE = CB + BA \) and \( F \) is on the extension of \( BA \) such that \( BF = BC + CA \). If \( AD \) is the diameter of the circumcircle of \( \triangle ABC \), prove that \( DI \perp EF \).
203. $ABCD$ is a parallelogram with diagonals $AC$, $BD$. Circle $\Gamma$ with diameter $AC$ cuts $DB$ at $P$, $Q$ and tangent line to $\Gamma$ through $C$ cuts $AB$, $AD$ at $X$, $Y$. Prove that points $P$, $Q$, $X$, $Y$ are concyclic.

204. Two triangles have a common inscribed in and circumscribed circle. Sides of one of them relate to the inscribed circle at the points $K$, $L$ and $M$, sides of another triangle at points $K_1$, $L_1$ and $M_1$. Prove that orthocentres of triangles $KLM$ and $K_1L_1M_1$ are match.

205. $ABCD$ is a convex quadrilateral with $\angle BAD = \angle DCB = 90^\circ$. Let $X$ and $Y$ be the reflections of $A$ and $B$ about $BD$ and $AC$ respectively. $P \equiv XC \cap BD$ and $Q \equiv DY \cap CA$. Show that $AC \perp PQ$.

206. In triangle $ABC$, $\angle A = 2\angle B = 4\angle C$. Prove that

$$\frac{1}{AB} = \frac{1}{BC} + \frac{1}{AC}$$

207. Point $P$ lies inside $\triangle ABC$ such that $\angle PBC = 70^\circ$, $\angle PCB = 40^\circ$, $\angle PBA = 10^\circ$ and $\angle PCA = 20^\circ$. Show that $AP \perp BC$.

208. The sides of a triangle are positive integers such that the greatest common divisor of any 2 sides is 1. Prove that no angle is twice of another angle in the triangle.

209. Two circles with centres $A$, $B$ intersect on points $M$, $N$. Radii $AP$ and $BQ$ are parallel (on opposite sides of $AB$). If the common external tangents meet $AB$ at $D$ and $PQ$ meet $AB$ at $C$, prove that $\angle CND$ is a right angle.

210. In an acute triangle $\triangle ABC$, the tangents to its circumcircle at $A$ and $C$ intersect at $D$, the tangents to its circumcircle at $C$ and $B$ and intersect at $E$. $AC$ and $BD$ meet at $R$ while $AE$ and $BC$ meet at $P$. Let $Q$ and $S$ be the mid-points of $AP$ and $BR$ respectively. Prove that $\angle ABQ = \angle BAS$.

211. Two circles $\Gamma_1$ and $\Gamma_2$ meet at $P$, $Q$. Their common external tangent (closer to $Q$) touches $\Gamma_1$ and $\Gamma_2$ at $A$, $B$. Line $PQ$ cuts $AB$ at $R$ and the perpendicular to $PQ$ through $Q$ cuts $AB$ at $C$. $CP$ cuts $\Gamma_1$ again at $D$ and the parallel to $AD$ through $B$ cuts $CP$ at $E$. Show that $RE \perp CD$.

212. Let $ABCD$ be a convex quadrilateral such that the angle bisectors of $\angle DAB$ and $\angle ADC$ intersect at $E$ on $BC$. Let $F$ be on $AD$ such that $\angle FED = 90^\circ - \angle DAE$. If $\angle FBE = \angle FDE$, prove that

$$EB^2 + EF \cdot ED = EB(EF + ED)$$

213. Let $ABC$ be a triangle. Let $P$ be a point inside such that $\angle BPC = 180^\circ - \angle ABC$ and $\frac{CP}{PT} = \frac{CB}{PA}$. Prove that $\angle APB = \angle CPB$. 

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214. Let $ABCD$ be a cyclic quadrilateral, and let $r_{XYZ}$ denote the inradius of $\triangle XYZ$. Prove that

$$r_{ABC} + r_{CDA} = r_{BCD} + r_{DAB}$$

215. $\triangle ABC$ is right-angled at $A$. $H$ is the projection of $A$ onto $BC$ and $I_1$, $I_2$ are the incenters of $\triangle AHB$ and $\triangle AHC$. Circumcircles of $\triangle ABC$ and $\triangle AI_1I_2$ intersect at $A, P$. Show that $AP, BC, I_1I_2$ concur.

216. An ant is crawling on the inside of a cube with side length 6. What is the shortest distance it has to travel to get from one corner to the opposite corner?

217. If $I_a$ is the excenter opposite to side $A$ and $O$ is the circumcenter of $\triangle ABC$. Then prove that:

$$(OI_a)^2 = R^2 + 2Rr_a$$

218. The two circles below have equal radii of 4 units each and the distance between their centers is 6 units. Find the area of the region formed by common points.

219. Triangle $ABC$ and its mirror reflection $A'B'C'$ are arbitrarily placed on a plane. Prove that the midpoints of the segments $AA'$, $BB'$ and $CC'$ lie on the same straight line.

220. The convex hexagon $ABCDEF$ is such that

$$\angle BCA = \angle DEC = \angle FAE = \angle AFB = \angle CBD = \angle EDF$$

Prove that $AB = CD = EF$.

221. Let $ABC$ be a triangle such that $BC = \sqrt{2}AC$. Let the line perpendicular to $AB$ passing through $C$ intersect the perpendicular bisector of $BC$ at $D$. Prove that $DA \perp AC$.

222. Three circles with centres $A, B, C$ touch each other mutually, say at points $X, Y, Z$. Tangents drawn at these points are concurrent (no need to prove that) at point $P$ such that $PX = 4$. Find the ratio of the product of radii to the sum of radii.

223. Hexagon $ABCDEF$ is inscribed in a circle of radius $R$ centered at $O$; let $AB = CD = EF = R$. Prove that the intersection points, other than $O$, of the pairs of circles circumscribed about $\triangle BOC$, $\triangle DOE$ and $\triangle FOA$ are the vertices of an equilateral triangle with side $R$.

224. Triangle $ABC$ has circumcenter $O$ and orthocenter $H$. Points $E$ and $F$ are chosen on the sides $AC$ and $AB$ such that $AE = AO$ and $AF = AH$. Prove that $EF = OA$. 
225. Let $AD$, $BE$, $CF$ be the altitudes of triangle $ABC$. Show that the triangle whose vertices are the orthocenters of triangles $AEF$, $BDF$, $CDE$ is congruent to triangle $DEF$.

226. Suppose $\ell_1$ and $\ell_2$ are parallel lines and that the circle $\Gamma$ touches both $\ell_1$ and $\ell_2$, the circle $\Gamma_1$ touches $\ell_1$ and $\Gamma$ externally in $A$ and $B$, respectively. Circle $\Gamma_2$ touches $\ell_2$ in $C$, $\Gamma$ externally in $D$ and $\Gamma_1$ externally at $E$. Prove that $AD$ and $BC$ intersect in the circumcenter of triangle $BDE$.

227. $\triangle ABC$ is scalene and $M$ is the midpoint of $BC$. Circle $\omega$ with diameter $AM$ cuts $AC$, $AB$ at $D$, $E$. Tangents to $\omega$ at $D$, $E$ meet at $T$. Prove that $TB = TC$.

228. Point $P$ lies inside triangle $ABC$ and $\angle ABP = \angle ACP$. On straight lines $AB$ and $AC$, points $C_1$ and $B_1$ are taken so that $BC_1 : CB_1 = CP : BP$. Prove that one of the diagonals of the parallelogram whose two sides lie on lines $BP$ and $CP$ and two other sides (or their extensions) pass through $B_1$ and $C_1$ is parallel to $BC$.

229. Let $ABC$ be a right angled triangle at $A$. $D$ is a point on $CB$. Let $M$ be the midpoint of $AD$. $CM$ intersects the perpendicular bisector of $AB$ at $E$. Prove that $BE \parallel DA$.

230. Prove that the pedal triangle of the Nine-point centre of a triangle with angles $75^\circ$, $75^\circ$, $30^\circ$ has to be equilateral.

231. $\triangle ABC$ is right-angled at $A$. $D$ and $E$ are the feet of the $A$-altitude and $A$-angle bisector. $I_1$, $I_2$ are the incenters of $\triangle ADB$ and $\triangle ADC$. Inner angle bisector of $\angle DAE$ cuts $BC$ and $I_1I_2$ at $K$, $P$. Prove that $PK : PA = \sqrt{2} - 1$.

232. In acute triangle $ABC$, there exists points $D$ and $E$ on sides $AC$, $AB$ respectively satisfying $\angle ADE = \angle ABC$. Let the angle bisector of $\angle A$ hit $BC$ at $K$. $P$ and $L$ are projections of $K$ and $A$ to $DE$, respectively, and $Q$ is the midpoint of $AL$. If the incenter of $\triangle ABC$ lies on the circumcircle of $\triangle ADE$, prove that $P$, $Q$, and the incenter of $\triangle ADE$ are collinear.

233. Let $(O)$ be the circumcircle $ABC$. $D$, $E$ lies on $(BC)$. $(U)$ touches to $AD$, $BD$ at $M$ and intouches $(O)$. $(V)$ touches to $AE$, $BE$ at $N$ and intouches $(O)$. $d$ touches external to $(U)$ and $(V)$. $P$ lie on $d$ and $d$ touches to the circumcircle of $BPC$. A circle touches to $d$ at $P$ and $BC$ at $H$. Prove $PH$ is the bisector of $\overline{MPN}$. $(BC)$ be circle with diameter $BC$.

234. In triangle $ABC$, the median through vertex $I$ is $m_i$, and the height through vertex $I$ is $h_{ii}$, for $I \in A, B, C$. Prove that if

\[
\left( \frac{h_a^2}{h_b h_c} \right)^{m_a} \left( \frac{h_b^2}{h_c h_a} \right)^{m_b} \left( \frac{h_c^2}{h_a h_b} \right)^{m_c} = 1
\]

then $ABC$ is equilateral.
235. Let $D$ and $E$ are points on sides $AB$ and $AC$ of a $\triangle ABC$ such that $DE \parallel BC$, and $P$ is a point in the interior of $\triangle ADE$, $PB$ and $PC$ meet $DE$ at $F$ and $G$ respectively. Let $O$ and $O'$ be the circumcenters of $\triangle PDG$ and $\triangle PFE$ respectively. Prove that $AP \perp OO'$.

236. Let $ABCD$ be a parallelogram. If $E \in AB$ and $F \in CD$, and provided that $AF \cap DE = X$, $BF \cap CE = Y$, $XY \cap AD = L$, $XY \cap BC = M$; show that $AL = CM$.

237. In a triangle $ABC$, $P$ is a point such that $\angle PBA = \angle PCA$. Let $B', C'$ be the feet of perpendiculars from $P$ onto $AB$ and $AC$. If $M$ is the midpoint of $BC$, the prove that $M$ lies on the perpendicular bisector of $B'C'$.

238. The lines joining the three vertices of triangle $ABC$ to a point in its plane cut the sides opposite vertices $A$, $B$, $C$ in the points $K$, $L$, $M$ respectively. A line through $M$ parallel to $KL$ cuts $BC$ at $V$ and $AK$ at $W$. Prove that $VM = MW$.

239. Let $ABCD$ be a parallelogram. Let $M \in AB$, $N \in BC$ and denote by $P$, $Q$, $R$ the midpoints of $DM$, $MN$, $ND$, respectively. Show that the lines $AP$, $BQ$, $CR$ are concurrent.

240. Let $(O_1)$, $(O_2)$ touch the circle $(O)$ internally at $M$, $N$. The internal common tangent of $(O_1)$ and $(O_2)$ cut $(O)$ at $E$, $F$, $R$, $S$. The external common tangent of $(O_1)$, $(O_2)$ cut $(O)$ at $A$, $B$. Prove that $AB \parallel EF$ or $AB \parallel SR$.

241. Let $H$ be the orthocenter of the triangle $ABC$. For a point $L$, denote the points $M$, $N$, $P$ are chosen on $BC$, $CA$, $AB$, respectively, such that $HM$, $HN$, $HP$ are perpendicular to $AL$, $BL$, $CL$, respectively. Prove that $M$, $N$, $P$ are collinear and $HL$ is perpendicular to $MP$.

242. The bisector of each angle of a triangle intersects the opposite side at a point equidistant from the midpoints of the other two sides of the triangle. Find all such triangles.

243. $ABCD$ trapezoid’s bases are $AB$, $CD$ with $CD = 2 \cdot AB$. There are $P$, $Q$ points on $AD$, $BC$ sides and $\frac{DP}{PA} = 2$, $\frac{BQ}{QC} = 3 : 4$. Find ratio of $ABQP$, $CDPQ$ quadrilaterals areas.

244. In convex quadrilateral $ABCD$ we found two points $K$ and $L$, lying on segments $AB$ and $BC$, respectively, such that $\angle ADK = \angle CDL$. Segments $AL$ and $CK$ intersects in $P$. Prove, that $\angle ADP = \angle BDC$.

245. Let $ABCD$ be a parallelogram and $P$ is a point inside such that $\angle PAB = \angle PCB$. Prove that $\angle PBC = \angle PDC$.  

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246. Consider a triangle $ABC$ and let $M$ be the midpoint of the side $BC$. Suppose $\angle MAC = \angle ABC$ and $\angle BAM = 105^\circ$. Find the measure of $\angle ABC$.

247. Let $AA_1$, $BB_1$, $CC_1$ be the altitudes of acute angled triangle $ABC$; $O_A$, $O_B$, $O_B$ are the incenters of triangles $AB_1C_1$, $BC_1A_1$, $CA_1B_1$, respectively; $T_A, T_B, T_C$ are the points of tangent of incircle of triangle $ABC$ with sides $BC, CA, AB$ respectively. Prove, that all sides of hexagon $T_AO_CT_BO_AT_CO_B$ are equal.

248. Let $ABC$ be a triangle and $P$ is a point inside. Let $AP$ intersect $BC$ at $D$. The line through $D$ parallel to $BP$ intersects the circumcircle of $\triangle ADB$ at $E$. The line through $D$ parallel to $CP$ intersects the circumcircle of $\triangle ADB$ at $F$. Let $X$ be a point on $DE$ and $Y$ is a point on $DF$ such that $\angle DCX = \angle BPD$ and $\angle DBY = \angle CPD$. Prove that $XY \parallel EF$.

249. Prove that if $N^*$, $O$ is the isogonal conjugate of the nine-point centre of $\triangle ABC$ and the circumcentre of $\triangle ABC$ respectively, then $A, N^*$, $M$ are collinear, where $M$ is the circumcentre of $\triangle BOC$.

250. So here’s easy one in using vectors. $ABCD$ is convex pentagon with $S$ area. Let $a, b, c, d, e$ are area of $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, $\triangle EAB$. Prove that:

$$S^2 - S(a + b + c + d + e) + ab + bc + cd + de + ea = 0$$

251. $ABC$ is a triangle with circumcentre $O$ and orthocentre $H$. $H_a, H_b, H_c$ are the foot of the altitudes from $A, B, C$ respectively. $A_1, A_2, A_3$ are the circumcentres of the triangles $BOC$, $COA$, $AOB$ respectively. Prove that $H_aA_1, H_bA_2, H_cA_3$ concur on the Euler’s line of triangle $ABC$.

252. The incircle $(I)$ of a given scalene triangle $ABC$ touches its sides $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$, respectively. Denote $\omega_B$, $\omega_C$ the incircles of quadrilaterals $BA_1IC_1$ and $CA_1IB_1$, respectively. Prove that the internal common tangent of $\omega_B$ and $\omega_C$ different from $IA_1$ passes through $A$.

253. Let $\omega_1, \omega_2$ be 2 circles externally tangent to a circle $\omega$ at $A, B$ respectively. Prove that $AB$ and the common external tangents of $\omega_1, \omega_2$ are concurrent.

254. Let $AC$ and $BD$ be two chords of a circle $\omega$ that intersect at $P$. A smaller circle $\omega_1$ is tangent to $\omega$ at $T$ and $AP$ and $DP$ at $E, F$ respectively. (Note that the circle $\omega_1$ will lie on the same side of $A, D$ with respect to $P$.) Prove that $TE$ bisects $ABC$ of $\omega$, and if $I$ is the incentre of $ACD$, show that $F = \omega_1 \cap EI \implies DF$ is tangent to $\omega_1$.

255. Assume that the point $H$ is the orthocenter of the given triangle $ABC$ and $P$ is an arbitrary point on the circumcircle of $ABC$. $E$ is a point on $AC$ such that $BE \perp AC$. Let us construct to parallelograms $PAQB$ and $PARC$. Assume that $AQ$ and $HR$ intersect at point $X$. Prove that $EX \parallel AP$.

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256. Let $AD, BE$ be the altitudes of triangle $ABC$ and let $H$ be the orthocenter. The bisector of the angle $DHC$ meets the bisector of the angle $B$ at $S$ and meet $AB, BC$ at $P, Q$, respectively. And the bisector of the angle $B$ meets the line $MH$ at $R$, where $M$ is the midpoint of $AC$. Show that $RPBQ$ is cyclic.

257. Prove that the Simson lines of diametrically opposite points on circumcircle of triangle $ABC$ intersect at nine point circle of the triangle.

258. In an equilateral triangle $ABC$. Prove that lines through $A$ that trisect outward semicircle on $BC$ as diameter trisect $BC$ as well.

259. Prove that the feet of the four perpendiculars dropped from a vertex of a triangle upon the four bisectors of the two other angles (two internal and two external angle bisectors) are collinear.

260. Let $ABC$ be a triangle. Let the angle bisector of $\angle A, \angle B$ intersect $BC, AC$ at $D, E$ respectively. Let $J$ be the incenter of $\triangle ACD$. Suppose that $EJDB$ is cyclic. Prove that $\angle CAB$ is equal to either $\angle CBA$ or $2\angle ACB$. 

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Solutions
