Problems on Functional Equations

The problem is always to find ALL functions with the given domain and range so that the given relation is satisfied; the notation \( f^2(\cdot) = f(\cdot)f(\cdot) \) is used. Extra conditions on the function(s) (monotonicity, continuity), when included in the statement of the problem, help to eliminate many potential solutions. **Warning.** The problems can be VERY hard!

I. Functions from rationals to rationals

1. \( f(xy) = f(x)f(y) = f(x + y) + 1; \)
2. \( f(x + y) = f(x) + f(y) + xy; \)

II. Functions from positive reals to positive reals

1. \( f(x)f(yf(x)) = f(x + y); \)
2. \( f(f(x)) + af(x) = b(a + b)x \) \((a, b \geq 0 \text{ if it helps}); \)
3. \( f(3x) \geq f(f(2x)) + x; \)
4. \( f(f(x) + y) = xf(1 + xy); \)
5. \( f^2(x) \geq f(x + y)f(x + y); \)
6. \( f(x)f(y) = f(xy) + f(x/y), f \text{ is continuous}; \)
7. \( f(f(f(x))) + 2x = f(3x), \lim_{x \to +\infty} (f(x) - x) = 0; \)

III. Functions from reals to reals

1. \( f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y); \)
2. \( f(xf(x) + f(y)) = y + f^2(x); \)
3. \( f(f(x)) = x^2 - 2; \)
4. \( f(x + g(y)) = xf(y) - yf(x) + g(x); \)
5. \( f(xy) = f(x)f(y) - f(x + y) + 1; \)
6. \( f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1; \)
7. \( f(x^2 + y^2 + 2f(xy)) = f^2(x + y); \)
8. \( f((x - y)^2) = f^2(x) - 2xf(y) + y^2; \)
9. \( f(x^2 - y^2) = xf(x) - yf(y); \)
10. \( f(x^2 - y^2) = (x - y)(f(x) + f(y)); \)
11. \( f(f(x) + y) = 2x + f(f(y) - x); \)
12. \( f(x^2 + f(y)) = y + f^2(x); \)
13. \( f(f(x) - y)) = f(x)f(y) - f(x) + f(y) - xy; \)
14. \( f(f(x) + y) = f(x^2 - y) + 4yf(x); \)
15. \( f(x + y) = f(x)f(y) - csin x \sin y, c > 1; \)
16. \( 3f(2x + 1) = f(x) + 5x, f \text{ is continuous}; \)
17. \( f(xy) = xf(y) + yf(x), f \text{ is continuous}; \)
18. \( f(x + y) = f(x) + f(y) + f(x)f(y), f \text{ is continuous}; \)
19. \( f(x + y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}, f \text{ is continuous}; \)
20. \( f(f(f(x))) = x, f \text{ is continuous}; \)
21. \( f(f(x)) - 2f(x) + x = 0, f \text{ is continuous}; \)
22. \( f(x) + f^{-1}(x) = 2x, f \text{ is strictly increasing, } f^{-1} \text{ is the inverse of } f; \)
23. \( f(x + f(y)) = f(x) + y^n, n = 1, 2, \ldots, f \text{ is monotone}; \)
24. \( f(xy) = yf(x), \lim_{x \to +\infty} f(x) = 0. \)